

MATRIX SOLUTION FOR INFLUENCE LINES

by

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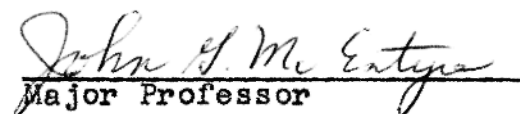
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# MATRIX SOLUTION FOR INFLUENCE LINES

By Bhalchandra S. Mehta<sup>1</sup>

## SYNOPSIS

The application of matrix algebra has made it possible for a structural engineer to analyze complex or highly redundant structures more easily, logically, and systematically. The problem of influence lines for portal frames is formulated in matrix form, using the principle of virtual work with displacements or forces as unknowns.

The application of the principle of virtual work is a convenient method to analyze a linear statically indeterminate structure. There are two main methods for analyzing it. The first method involves determining certain redundant forces or moments by solving the elastic compatibility equations. All elastic characteristics of the structure are contained in a flexibility matrix which consists of an ordered array of the flexibility influence coefficients. This method is known as the "force method" of analysis. In the second method of analysis, which is known as the "displacement method", joint rotations or displacements are considered as unknowns and a solution of the simultaneous joint equilibrium equations is to be performed. These equations relate the redundant forces in terms of the assumed deflections. They are represented by the stiffness matrix

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composed of an ordered array of the stiffness influence coefficients. Civil engineering structures can be analyzed conveniently by the stiffness matrix technique.

## INTRODUCTION

The matrix form of the equations of plane redundant structures by the displacement and force methods is based upon ideas developed and discussed by J. H. Argyris (1)<sup>2</sup> and by making an adaptation from the equations deduced by that author for plane structures composed of rigid joints and straight bars with constant transverse section. As the matrix formulation of structural theory was pioneered and was given a very thorough and effective treatment by Argyris, the matrix equations are sometimes known as Argyris matrix equations for analyzing of stresses and deflections of various types of beam and frame systems.

The application of a matrix procedure to the analysis of structures consisting of flexural members is presented herein. The usual and standard procedures of structural analysis, such as the virtual work method or dummy unit load method, provide very convenient means for evaluating deflections or stresses under static conditions (2). The method presented herein consists of a matrix systematizing of the virtual work procedure. The numerical operations which are done correspond exactly to those which are

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<sup>2</sup>Numerals in parentheses, thus (1), refer to corresponding items in the Reading References.

usually performed in the standard virtual work solution. The direct results of usual matrix procedure are two influence coefficient matrices. The first of these, the stress matrix, represents the forces in each of the members of the structure due to successive applications of unit values of the external loads, while the second, the flexibility matrix, represents the deflections at the points of loading due to unit values of the external loads. When these two matrices have been evaluated, it is a simple operation to obtain stresses or deflections due to any system of externally applied concentrated loads.

Matrix methods are convenient methods to solve linear simultaneous equations. As the equations are solved by matrix methods, the final results, that is, the different ordinates, required for drawing influence lines corresponding to the different positions of moving load, are in the form of a matrix.

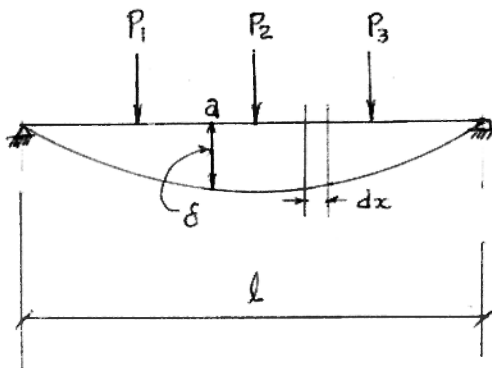
The development of the electronic digital computer has helped in analyzing highly indeterminate structures which previously had been difficult and laborious to analyze by the hand method of calculation (3). The application of matrix theory to structural analysis thus becomes more logical and permits the concise formulation of large problems and control of the data required for feeding to the electronic digital computer. It has therefore stimulated an intense interest in the use of matrix methods to formulate structural theory.

## VIRTUAL WORK

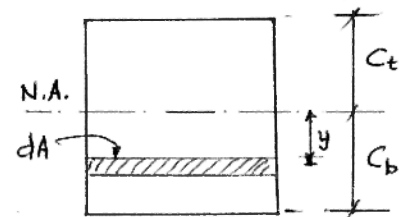
The most general and direct method for computing the deflections of structures is the method of virtual work. The principle of virtual displacement is used to develop the basis for the method of virtual work for computing the deflection of a structure. The theorem of virtual work may be stated as follows:

"If a body which is in equilibrium under a system of external loads is given any small (virtual) deformation, then the work done by the external loads during this virtual deformation is equal to the increase in internal strain energy stored in the body." (4)

Consider the beam of Fig. 1. It is desired to know the deflection at point "a" in the beam caused by the external loads



(a) Beam



(b) Cross section

Fig. 1. VIRTUAL WORK.

$P_1$ ,  $P_2$ , and  $P_3$ . If the loads were removed from the beam and a unit load placed at "a", small stresses and deformations would be developed in the fibers of the beam, and a small deflection would occur at "a". The external loads are now replaced on the beam. Due to these loads the fiber stresses and deformations will be increased, and the unit load at "a" would deflect an additional amount  $\delta$ . The internal work performed by the unit load stresses, as they are carried through the additional fiber deformations, equals the external work performed by the unit load as it is carried through the additional deflection  $\delta$  (5). By using the flexural formula,

$$\text{Unit stress in } dA = \frac{my}{I}$$

where  $m$  is the moment at any section due to the unit load.

$$\text{Total stress in } dA = \frac{my}{I} dA$$

When the external loads are returned to the structure,

$$\text{Deformation of } dx \text{ length} = \epsilon dx$$

$$= \frac{f}{E} dx$$

$$= \frac{My}{EI} dx$$

where  $M$  is the moment at any section in the beam due to the external loads.

$$\text{Work done in } dA = (\text{total stress in } dA)$$

$$(\text{deformation of } dx \text{ length})$$

$$\begin{aligned}
 &= \left( \frac{my}{I} dA \right) \left( \frac{My}{EI} dx \right) \\
 &= \frac{Mmy^2}{EI^2} dA dx
 \end{aligned}$$

The total work performed on the cross section is expressed by

$$\int_{c_b}^{c_t} \frac{Mmy^2}{EI^2} dA dx = \frac{Mm}{EI^2} \int_{c_b}^{c_t} y^2 dA dx \quad (1)$$

Since the expression  $\int y^2 dA$  is the moment of inertia of the section, equation (1) becomes

$$\text{Work} = \frac{Mm}{EI} dx \quad (2)$$

By integrating from 0 to  $\ell$  of this expression, the internal work performed in the entire beam can be obtained. Then equation (2) becomes

$$W_1 = \int_0^{\ell} \frac{Mm}{EI} dx$$

The external work performed by the unit load as it is carried through the distance  $\delta$  is  $1 \times \delta$ . By applying the law of conservation of energy, namely,

"If a structure and the external loads acting on it are isolated so that these neither receive nor give out energy, then the total energy of this system remains constant." (4)

This implies that for a body in static equilibrium, the variation in internal energy must equal to the variation in external energy. Therefore the following expression may be written.



$$\begin{aligned}
 W_e &= W_i \\
 1 \times \delta &= \int_0^l \frac{Mm}{EI} dx \\
 \delta &= \int_0^l \frac{Mm}{EI} dx \quad (3)
 \end{aligned}$$

It should be noted that the derivation for the theorem of virtual work is based entirely on energy principles. This implies that the principle of virtual work is applicable to non-Hookean materials as well as materials which are linearly elastic.

It will be seen that this concept of energy principles is readily adaptable to matrix formulation and analysis.

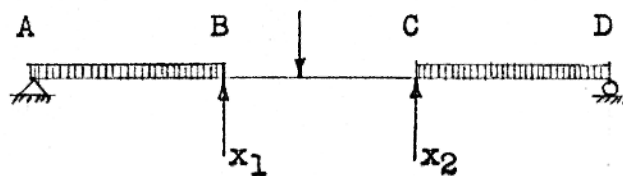
#### INFLUENCE COEFFICIENTS

Some systematic order of computations is adopted in many methods of analyzing a statically indeterminate structure. First, the redundants and corresponding statically determinate primary structure are selected. These redundants are considered as forces and/or moments. They are computed by solving an equal number of simultaneous equations. Each equation expresses a known deflection condition for the primary structure in terms of the redundants. Once the redundants are known, then the stresses and deflections for the entire structure may be computed as in the case of a statically determinate structure. Such a method of structural analysis is referred to as a force method of structural analysis (6).

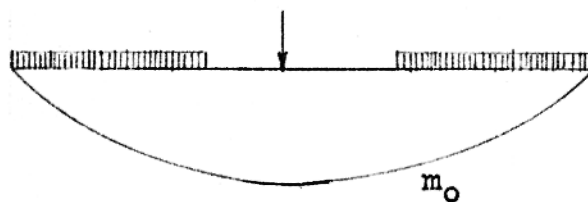
Consider a three-span continuous beam on rigid supports, as shown in Fig. 2(a). Assume the temperature of the material remains constant. This beam is statically indeterminate to the second degree. The intermediate support reactions B and C are chosen as redundants  $x_1$  and  $x_2$ . The structure is made statically determinate by removing the two intermediate supports.

As only flexural strain energy is considered, the total bending moment  $M$  will be all that is required to find the strain energy of the statically indeterminate structure. By the principle of superposition, the total moment can be considered as composed of three parts: (a) The moment " $m_0$ " due to the applied loads only acting on the released structure (residual structure), that is,  $x_1 = 0$  and  $x_2 = 0$ , Fig. 2(b). Call this the "particular solution" of this problem. It satisfies the conditions of equilibrium but not the boundary conditions of the problem. In this case the continuity of the intermediate supports are the boundary conditions. (b) The moment " $m_1x_1$ " due to the action of the redundant  $x_1$  alone on the residual structure, that is, the loads are removed and  $x_2 = 0$ . Here  $m_1$  is the moment due to  $x_1 = 1$  acting along. See Fig. 2(c). (c) The moment " $m_2x_2$ " due to the action of the redundant  $x_2$  alone on the residual structure, that is, the loads are removed and  $x_1 = 0$ . Here  $m_2$  is the moment due to  $x_2 = 1$  acting alone. See Fig. 2(d).

Call the parts (b) and (c) "complementary functions". These represent the effects of the redundants whose function is to satisfy the boundary conditions (7). Thus the following equation for strain energy " $U$ " may be written.



(a) Beam



(b) Moment Due to Load

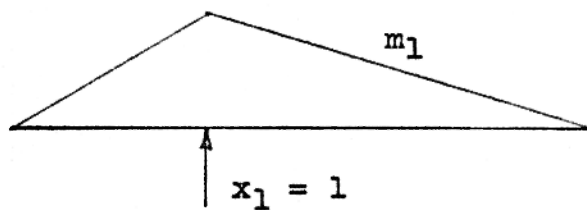
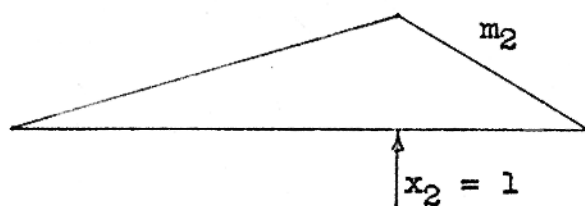
(c) Moment Due to  $x_1 = 1$ (d) Moment Due to  $x_2 = 1$ 

Fig. 2. MOMENT DIAGRAMS DUE TO APPLIED LOAD AND UNIT VALUES OF REDUNDANTS.

$$U = \int_s \frac{M^2}{2EI} ds \quad (4)$$

with the region of integration extended over the whole length of the beam. The equation (4) may be rewritten by putting the moment  $M$  in terms of its components so that

$$U = \int_s \frac{1}{2EI} (m_0 + m_1 x_1 + m_2 x_2)^2 ds \quad (5)$$

Applying the theorem of least work,  $\frac{\partial U}{\partial x} = 0$ , the two equations for the determination of  $x_1$  and  $x_2$  can be computed.

$$\begin{aligned} \frac{\partial U}{\partial x_1} &= \int_s \frac{\partial}{\partial x_1} \left( \frac{M^2}{2EI} \right) ds = 0 \\ &= \int_s \frac{\partial}{\partial x_1} \frac{1}{2EI} (m_0 + m_1 x_1 + m_2 x_2)^2 ds = 0 \\ &= \int_s \frac{m_1}{EI} (m_0 + m_1 x_1 + m_2 x_2) ds = 0 \end{aligned} \quad (6)$$

and similarly

$$\frac{\partial U}{\partial x_2} = \int_s \frac{m_2}{EI} (m_0 + m_1 x_1 + m_2 x_2) ds = 0 \quad (7)$$

These two equations (6) and (7) may be expanded as follows:

$$x_1 \int_s \frac{m_1^2}{EI} ds + x_2 \int_s \frac{m_1 m_2}{EI} ds + \int_s \frac{m_1 m_0}{EI} ds = 0 \quad (8a)$$

$$x_1 \int_s \frac{m_2 m_1}{EI} ds + x_2 \int_s \frac{m_2^2}{EI} ds + \int_s \frac{m_2 m_0}{EI} ds = 0 \quad (8b)$$

Considering now Castigliano's second theorem, namely,

$$\frac{\partial U}{\partial x} = \delta$$

and applying it to the deflection  $\delta_1$  at the position and in the direction of  $x_1$ ,

$$\delta_1 = \frac{\partial U}{\partial x_1} = x_1 \int_s \frac{m_1^2}{EI} ds + x_2 \int_s \frac{m_1 m_2}{EI} ds + \int_s \frac{m_1 m_0}{EI} ds \quad (9)$$

If the two redundants are zero, that is,  $x_1 = 0$  and  $x_2 = 0$ , equation (9) becomes

$$\delta_1 = \int_s \frac{m_1 m_0}{EI} ds = f_{10}$$

where  $f_{10}$  is the deflection of the released structure at the position and in the direction of  $x_1$  due to the applied loads (Fig. 3(a)). If the applied load is removed,  $m_0 = 0$ , and the redundant  $x_2 = 0$ . Then equation (9) becomes

$$\delta_1 = x_1 \int_s \frac{m_1^2}{EI} ds = x_1 f_{11}$$

where  $f_{11}$  is the deflection of the released structure at the position and in the direction of  $x_1$  for a unit value of  $x_1$  acting alone (Fig. 3(b)).

If the applied load is removed, then  $m_0 = 0$ , and the redundant  $x_1 = 0$ . Then equation (9) becomes

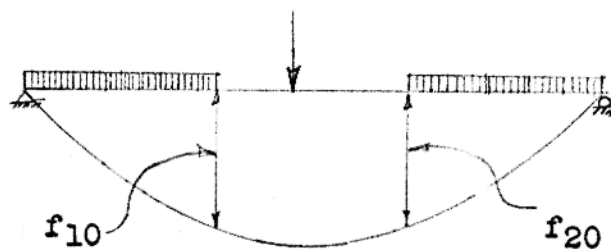
$$\delta_1 = x_2 \int_s \frac{m_1 m_2}{EI} ds = x_2 f_{12}$$

where  $f_{12}$  is the deflection of the released structure at the position and in the direction of  $x_1$  for a unit value of  $x_2$  acting alone (Fig. 3(c)).

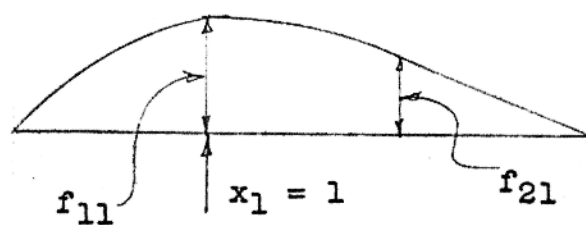
Equation (8a) can be rewritten in the form

$$f_{11} x_1 + f_{12} x_2 + f_{10} = 0 \quad (10)$$

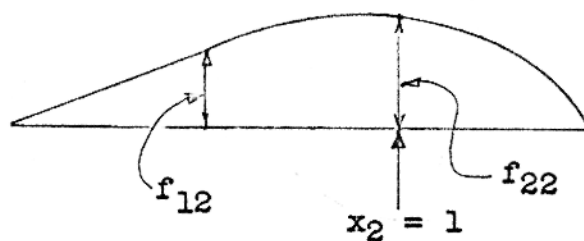
where it is now seen that it is an expression of the fact that



(a) Deflection Due to Load



(b) Deflection Due to  $x_1 = 1$



(c) Deflection Due to  $x_2 = 1$

Fig. 3. DEFLECTION DUE TO APPLIED LOAD AND REDUNDANTS.

the displacements due to the several separate effects must sum to zero to produce the required boundary condition of a rigid support B (Fig. 2(a)). Similarly, equation (8b), which relates to the support C, can be rewritten in the form

$$\sqrt{f_{21} x_1 + f_{22} x_2 + f_{20} = 0} \quad (11)$$

where

$f_{21}$  = the deflection of the released structure at the position and in the direction of  $x_2$  for a unit value of  $x_1$  acting alone

$f_{22}$  = the deflection of the released structure at the position and in the direction of  $x_2$  for a unit value of  $x_2$  acting alone

$f_{20}$  = the deflection of the released structure at the position and in the direction of  $x_2$  due to the applied loads.

The equations (10) and (11) are alternative arrangements of equations (6) and (7) and are equivalent in all respects. It is to be noted that  $f_{12} = f_{21}$ . This follows from the forms of  $f_{12}$  and  $f_{21}$ ,

$$f_{12} = \int_s \frac{m_1 m_2}{EI} ds = \int_s \frac{m_2 m_1}{EI} ds = f_{21}$$

and also from Maxwell's law of reciprocal deflections, which states:

"In any structure the material of which is elastic and follows Hooke's law and in which the supports are unyielding and the temperature constant, the deflection at one point A in a structure due to a load applied at

another point B is exactly the same as the deflection at B if the same load is applied at A." (5, 6)

The example, discussed above, is one in which there are two redundants,  $x_1$  and  $x_2$ . In general the solution of a structure with  $n$  redundants will lead to a set of  $n$  simultaneous linear algebraic equations in which there will be  $n^2$  terms involving influence coefficients of the type  $f_{ij}$  ( $i, j = 1, 2, \dots, n$ ) and  $n$  quantities of the type  $f_{i0}$  ( $i, 0 = 1, 2, \dots, n$ ), where  $f$ 's and  $f_0$ 's are formed in exactly the same way as discussed in the above example. Thus

$$f_{ij} = \int_s \frac{m_i m_j}{EI} ds \quad (12)$$

and

$$f_{i0} = \int_s \frac{m_i m_0}{EI} ds \quad (13)$$

where  $f_{ij}$  = the deflection at point  $i$ , in the direction of load  $i$ , due to unit load at  $j$ , with all other forces removed

$f_{i0}$  = the deflection at point  $i$ , in the direction of load  $i$ , due to the external loads applied.

In the general case, the meanings of the  $f$ 's and  $f_0$ 's remain as displacements which may include deflections and rotations according to whether the corresponding redundants are forces or moments. The equations to be solved will then be



$$\begin{aligned}
f_{11}x_1 + f_{12}x_2 + \dots f_{1j}x_j + \dots f_{1n}x_n + f_{10} &= 0 \\
f_{21}x_1 + f_{22}x_2 + \dots f_{2j}x_j + \dots f_{2n}x_n + f_{20} &= 0 \\
\dots &\dots \\
f_{i1}x_1 + f_{i2}x_2 + \dots f_{ij}x_j + \dots f_{in}x_n + f_{i0} &= 0 \\
\dots &\dots \\
f_{n1}x_1 + f_{n2}x_2 + \dots f_{nj}x_j + \dots f_{nn}x_n + f_{n0} &= 0
\end{aligned} \tag{14}$$

The matrix formulation of equation (14) gives

$$\begin{bmatrix} f_{11} & f_{12} & \dots & f_{1j} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2j} & \dots & f_{2n} \\ \vdots & & \dots & \vdots & \dots & \vdots \\ f_{i1} & f_{i2} & \dots & f_{ij} & \dots & f_{in} \\ \vdots & & \dots & \vdots & \dots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nj} & \dots & f_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_{10} \\ f_{20} \\ \vdots \\ f_{i0} \\ \vdots \\ f_{n0} \end{bmatrix} = 0 \tag{15}$$

or in the generalized form as

$$F X + F_0 = 0$$

The matrix  $F$  in the above equation is commonly referred to as the "flexibility" matrix. It is seen that this matrix is made up of the influence deflection coefficients  $f_{ij}$  for the structure.  $X$  is a column vector, the elements of which are the redundant forces or moments. The column vector  $F_0$  is made up of the displacements and rotations computed for the determinate structure (Fig. 2(b)) caused by the external loading.

It is easily observed that the matrix  $F$  is a square matrix and also a symmetric one. Therefore the inverse of  $F$  exists.

The solution can be written in the form

$$X = -F^{-1}F_0$$

where  $F^{-1}$  is the inverse of flexibility matrix  $F$  and is known as "stiffness" matrix. It will be discussed later.

In other cases of statically indeterminate structures, the order of computation discussed in a force method is completely inverted. Such a method of analysis is called a displacement method of structural analysis. In this method, first the internal forces and couples are expressed in terms of the key displacement components of the structure. Such expressions are substituted into the key equilibrium equations of the structures. A system of linear simultaneous equations involving the key displacements as the unknowns can be obtained. The values of the displacements obtained from the solution of these equations are then substituted into the original expressions for the internal forces and couples to obtain the values of the latter. Once all the internal forces and couples are known, it is possible to compute the reactions of the structure.

Consider any structure which is loaded by the forces  $p_1, p_2, \dots, p_1, \dots, p_n$  (3). The temperature of the material remains constant and the supports are rigid. Applying Castigliano's First Theorem,

$$p_1 = \frac{\partial U}{\partial y_1} \quad (16)$$

where  $U$  is the strain energy stored within a structure, and  $y_1$  is the deflection of the point of application of the load  $p_1$  in the direction of  $p_1$ .

Equation (16) may be expanded if the strain energy is evaluated in terms of the loads  $p_1$ . It may be written as

$$\begin{aligned}
p_1 &= \frac{\partial U}{\partial y_1} \\
&= \frac{\partial U}{\partial p_1} \cdot \frac{\partial p_1}{\partial y_1} + \frac{\partial U}{\partial p_2} \cdot \frac{\partial p_2}{\partial y_1} \dots + \frac{\partial U}{\partial p_1} \cdot \frac{\partial p_1}{\partial y_1} \\
&\quad + \dots + \frac{\partial U}{\partial p_n} \cdot \frac{\partial p_n}{\partial y_1} \\
&= \sum_{j=1,2,\dots,n} \left( \frac{\partial U}{\partial p_j} \right) \left( \frac{\partial p_j}{\partial y_1} \right) \tag{17}
\end{aligned}$$

From Castigliano's Second Theorem,

$$y_j = \frac{\partial U}{\partial p_j} \tag{18}$$

Substituting equation (18) into equation (17),

$$p_1 = \sum_{j=1,2,\dots,n} y_j \left( \frac{\partial p_j}{\partial y_1} \right) \tag{19}$$

The partial derivative  $\frac{\partial p_j}{\partial y_1}$  represents the force developed

at point  $j$  due to a unit deflection of point  $1$ , all other points remaining fixed. This force is represented by the symbol  $s_{j1}$ .

The subscript  $j$  represents the point at which the force acts and the subscript  $1$  the point at which the unit deflection is imposed.

Equation (19) then becomes

$$p_1 = \sum_{j=1,2,\dots,n} y_j s_{j1} \tag{20}$$

By Maxwell's law of reciprocal deflections,

$$s_{j1} = s_{1j}$$

and hence

$$p_1 = \sum_{j=1,2,\dots,n} y_j s_{1j} \tag{21}$$

Equation (21) may be written for different loads in a system of simultaneous equations.

$$\begin{aligned}
 p_1 &= s_{11}y_1 + s_{12}y_2 + \dots + s_{1n}y_n \\
 p_2 &= s_{21}y_1 + s_{22}y_2 + \dots + s_{2n}y_n \\
 &\vdots \\
 p_n &= s_{n1}y_1 + s_{n2}y_2 + \dots + s_{nn}y_n
 \end{aligned}
 \tag{22}$$

Matrix formulation of equation (22) gives

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ & & & \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}
 \tag{23}$$

or in general form

$$P = S Y$$

where  $P$  is a column vector made up of the load components  $p_1, p_2, \dots, p_1, \dots, p_n$ .  $Y$  is also a column vector but it consists of the deflection components  $y_1, y_2, \dots, y_1, \dots, y_n$ .  $S$  is a square matrix consisting of an ordered array of the stiffness influence coefficients  $s_{ij}$  of the structure. Matrix  $S$  is referred to as the "stiffness" matrix of the structure. As  $S$  is a symmetrical matrix, its inverse exists.

The solution may be written in the form

$$Y = S^{-1}P$$

where  $S^{-1}$  is the inverse of stiffness matrix  $S$  and is the flexibility matrix  $F$  as shown before.

The following relationship between the two matrices  $F$  and  $S$  may be shown.

$$F = S^{-1} \text{ and } S = F^{-1}$$

That is, the stiffness matrix is the inverse of the flexibility matrix, and vice versa. It has been proved also that if the product of these two matrices  $F$  and  $S$  is formed, the identity matrix can be obtained (4).

### INFLUENCE LINES

Many structures are subjected to moving loads. It should therefore be clear that it is essential for a structural analyst to understand the methods by which the position of live load which causes the maximum stress at any point may be determined. This may be done conveniently by means of diagrams or curves that show the effect of moving a unit load across the structure. Such curves are commonly known as "influence lines" (8).

An influence line, then, may be defined as a curve which shows for a particular section or point the variation in shear, moment, reaction, or other direct function due to a unit load moving across the structure. For a particular function it can therefore be constructed by placing a unit load at various points on a structure. For each of these positions of the unit load, the value of the function at a particular section of the structure can be determined. Thus an influence line for reaction, shear, or moment shows the variation of reaction, shear, or moment at a particular section of the structure due to a unit load placed

anywhere along the structure.

Influence lines may be used for two very important purposes:

1. To determine what position of live loads will lead to a maximum value of the particular function for which an influence line has been constructed.

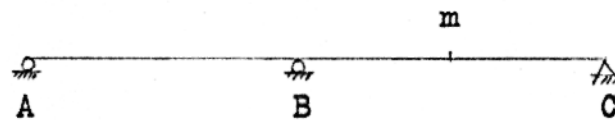
2. To compute the value of that function with the loads so placed or for any loading condition (6).

In plotting the influence lines for a statically determinate structure, the resulting diagrams are composed of straight-line segments. In this case, the ordinates for a few controlling points are computed, and these values are connected with a set of straight lines. Influence lines for indeterminate structures are not as simple to draw as they are for determinate structures. In this case, the computation of ordinates at a large number of points is required because the diagrams are either curved or made up of a series of chords.

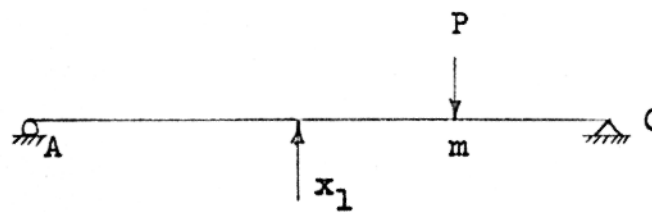
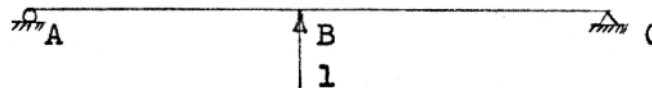
The problem of preparing the diagrams is not as difficult as it might first appear. A large percentage of the work may be eliminated by applying Maxwell's law of reciprocal deflections, as discussed before.

Influence lines obtained are sketched by two methods, namely, (1) quantitatively, and (2) qualitatively. Influence lines obtained by actual computation of ordinates are said to be quantitative influence lines.

Consider a beam shown in Fig. 4(a). It is statically indeterminate to first degree. It is required to find the influence line for  $R_b$ .



(a) Given Beam

(b) Redundant  $R_b$  Removed(c) Virtual Beam to Solve for  $\delta_{11}$ 'sFig. 4. INFLUENCE SOLUTION FOR  $R_b$ .

Let  $X_1 = R_b$

then  $P \delta_{1m} + X_1 \delta_{11} = 0$

$$X_1 = - \frac{P \delta_{1m}}{\delta_{11}} \quad (24)$$

If  $P = 1$ , equation (24) is the equation for the ordinate of the influenced line at point "m", or

$$X_1 = - \frac{\delta_{1m}}{\delta_{11}} \quad (25)$$

Considering Maxwell's law,

$$\delta_{1m} = \delta_{m1}$$

then equation (25) becomes

$$X_1 = - \frac{\delta_{m1}}{\delta_{11}}$$

It is now evident that the unit load need only be placed at B, and the deflections at various points across the beam are then computed. Dividing each of these values by  $\delta_{11}$  gives the ordinates for the influence line. The deflections at various points may be computed by the conjugate beam method or any other method.

Influence lines obtained by mere sketching are said to be qualitative influence lines. Muller-Breslau's principle is conveniently applied in determining the qualitative influence lines. This principle may be stated as follows.

"The deflected shape of a structure represents the influence line for a function such as stress, shear, moment, or reaction component if the function is allowed to act through a unit distance." (5)

This principle is applicable to both statically determinate and



indeterminate beams, frames, and trusses. It is thus possible to roughly sketch the diagram with sufficient accuracy to locate the critical positions for live load for various functions of the structure.

## DISCUSSION

The two general methods of compatibility, leading to a flexibility matrix, and equilibrium, leading to a stiffness matrix, form the framework of the discussion of this report.

The first method consists of removing the redundant reactions or internal forces and finding the amount by which the compatibility conditions (which express the continuity of the structure) are violated under the action of external load. Next, the effects of indeterminate reactions on this displacement or rotation differences are found. The final step is the determination of the indeterminate forces or moments from a set of simultaneous equations which express the conditions of compatibility. At every step in the procedure, the equilibrium conditions have been satisfied and the final set of equations insures that the compatibility relations also hold. The number of simultaneous equations to solve is, in general, the number of redundancies. Thus for the frame shown in Fig. 5(a), the four statically determinate structures, namely, 5 (b, c, d, and e), may be analyzed by finding in each case the horizontal and vertical displacements and the rotation of the base of the right column of the free end. Then three simultaneous equations, which have  $H$ ,  $V$ , and  $M$  as

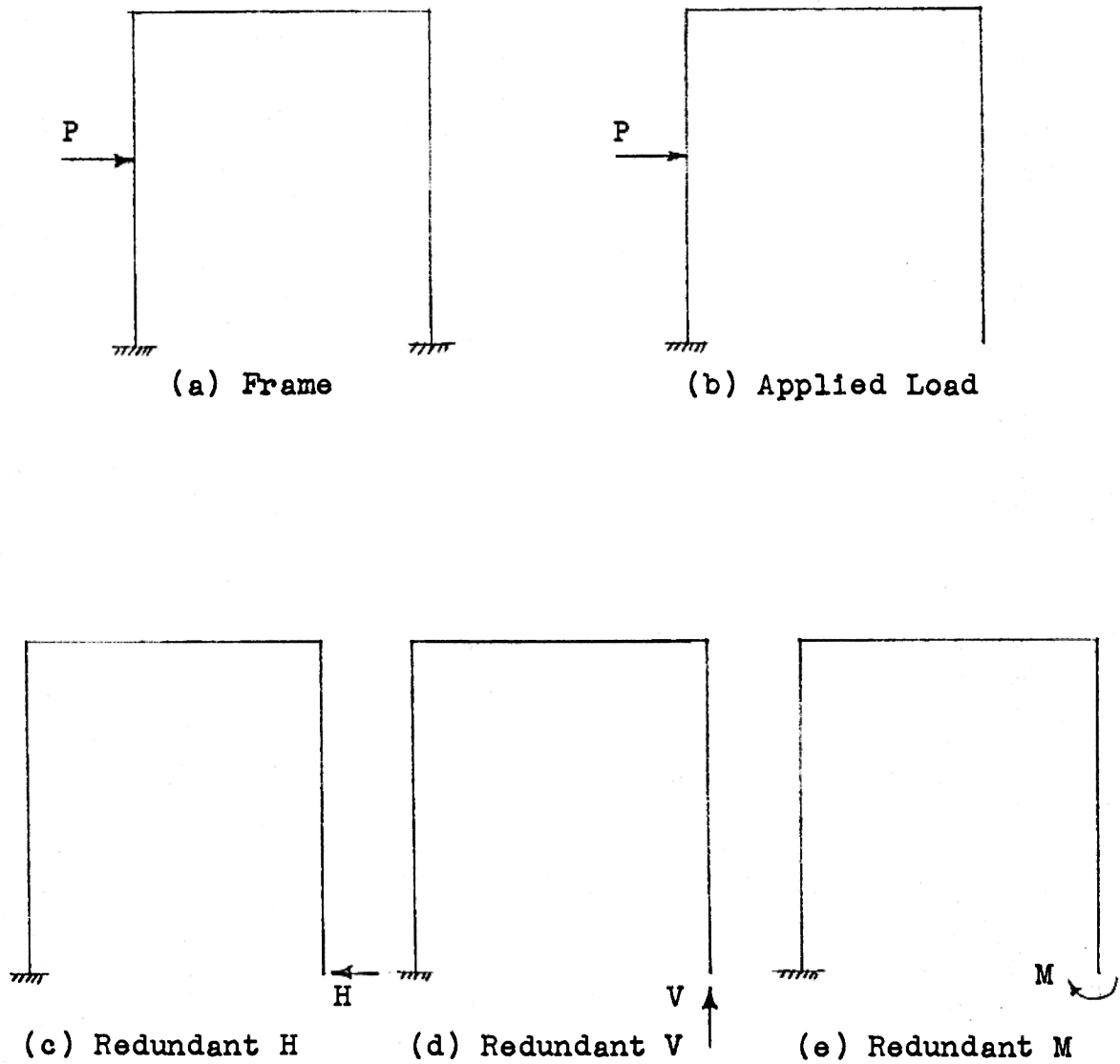


Fig. 5. FRAME WITH APPLIED LOAD AND REDUNDANTS.

unknowns, may be set up to express the conditions of fixity of the base of the right column.

The name usually given to this general method of attack is the method of compatibility, because the equations which are solved are compatibility equations. In the methods of compatibility, the unknowns are the indeterminate forces and moments. The matrix form of the simultaneous equations which arise in solutions by the method of compatibility is

$$AF = X_0$$

where  $A$  is the matrix of the structure, and  $F$  is a column vector, the elements of which are the redundant forces or moments. The column vector  $X_0$  is made up of the displacements and rotations computed for the determinate structure (Fig. 6(b)) caused by the external loading. Since the equations have a solution the determinant of the matrix  $A$  does not vanish, and hence its inverse exists. If the matrix  $A^{-1}$  is computed, one can find the indeterminate reactions immediately for any loading of the structure merely by solving part of Fig. 6(b) again and forming the matrix product  $A^{-1}X_0$ . The symmetry of the matrix  $A$  follows from the reciprocal theorem. It can be shown that the inverse of a symmetric matrix is symmetric, or  $A^{-1}$  can be shown to be symmetric directly in this case.

In this method, the symmetric matrix  $A$  is usually called the flexibility matrix of the structure (actually a flexibility matrix for the structure, as there are many choices of unknowns). The reason for the name is not hard to explain. Consider

$$AF = X_0$$

If all the elements of  $A$  were multiplied by a number greater than unity, the displacement vector  $X_0$  would be large for the same set of forces  $F$ ; in other words, the 'size' of  $A$  (in some vague sense) is a measure of the flexibility of the structure. By doubling  $A$ , one describes a structure which is twice as flexible as the original one.

In the second general method of analysis in which the stiffness matrix is used, constraints are added instead of being removed. The 'slope-deflection' method of analysis is an example of this method (9). For the same frame in Fig. 6(a), fixed-end moments are found in Fig. 6(b) due to the load on the left column. Then fixed-end moments are found in Fig. 6 (c, d, and e) due to the imposed rotations  $\theta_1$  and  $\theta_2$  and the side sway  $\Delta$ . The equations which are written are equations of moment equilibrium of the joints at the top of the frame and of horizontal equilibrium of the beam. The unknowns in these equations are values of the joint rotations and the side sway. It is reasonable to call this procedure the method of equilibrium since the equations which are solved are equations of equilibrium. The compatibility relations are automatically satisfied at every step. In this case the number of simultaneous equations is the same by both methods.

In general this is not so, and one method or the other is often better suited to a specific problem. By this method, the equations are of form

$$BX = F_0$$

$X$  being the vector of the joint rotations and sway, and  $F_0$  the

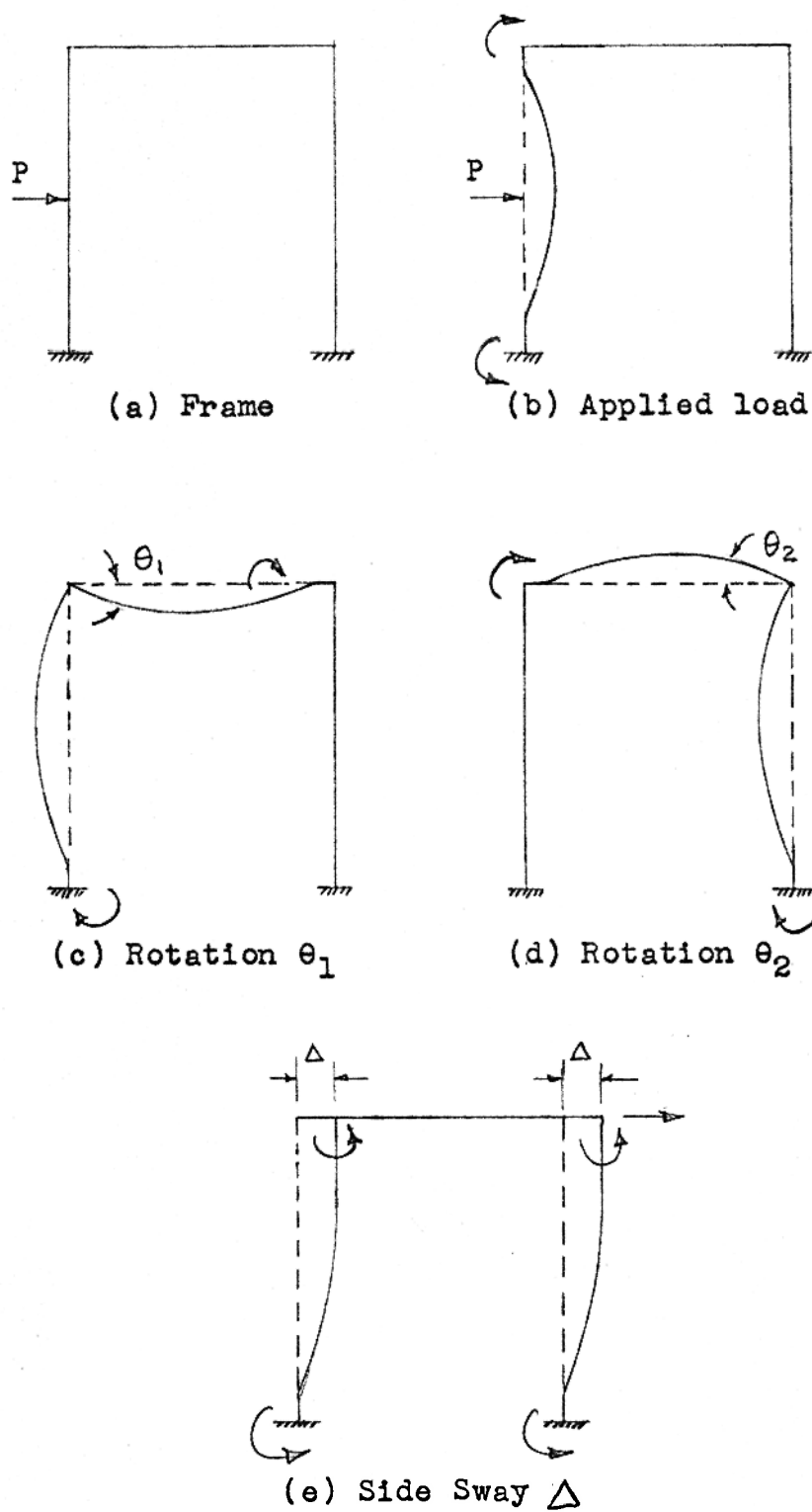


Fig. 6. FRAME WITH APPLIED LOAD, ROTATIONS, AND SIDE SWAY.

vector whose elements are fixed-end moments or horizontal forces on the beam (none in this case). The set of equations can be inverted and given a similar interpretation to  $B^{-1}$ . Both  $B$  and  $B^{-1}$  are symmetric. It is natural to call the matrix  $B$  a stiffness matrix of the structure, for by doubling  $B$  a structure becomes twice as stiff as the original one.

In recent years the matrix formulation of static structural problems has been encouraged by the presence of digital computers in many design offices.

### EXAMPLE PROBLEMS

#### Force Method Example

Given: A frame as shown in Fig. 7(a).

Required: Influence lines for  $H_D$ ,  $V_D$ , and  $M_D$ .

The given frame is statically indeterminate to the third degree. It is made determinate and stable by removing the fixed-end  $D$ , that is, making the end  $D$  free. The redundants will be  $x_1$ ,  $x_2$ , and  $x_3$ , as shown in Fig. 7 (b, c, and d). At this stage, a matrix of the structure may be defined as

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

This matrix is independent of the load system on the structure but depends upon the redundants chosen and the dimension of the structure. Due to this, a work for the calculation of

redundants for various load positions on the structure is reduced considerably.

For the load position marked as (1) in Fig. 7(a), the elastic equations are

$$f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{10}^{(1)} = 0$$

$$f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{20}^{(1)} = 0$$

$$f_{31}x_1 + f_{32}x_2 + f_{33}x_3 + f_{30}^{(1)} = 0$$

For the load position marked as (2) in Fig. 7(a), the elastic equations are

$$f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{10}^{(2)} = 0$$

$$f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{20}^{(2)} = 0$$

$$f_{31}x_1 + f_{32}x_2 + f_{33}x_3 + f_{30}^{(2)} = 0$$

Here it is seen that the matrix of the given structure does not change with the position of one kip load, but it is the same for all positions of the load. It is to be noted carefully that the constant column vectors  $F_{10}$  do change. With the one kip load in positions (3), (4), ..., to (16), the column vectors will be

$$\begin{bmatrix} f_{10}^{(3)} \\ f_{20}^{(3)} \\ f_{30}^{(3)} \end{bmatrix}, \begin{bmatrix} f_{10}^{(4)} \\ f_{20}^{(4)} \\ f_{30}^{(4)} \end{bmatrix}, \dots, \begin{bmatrix} f_{10}^{(16)} \\ f_{20}^{(16)} \\ f_{30}^{(16)} \end{bmatrix}$$

Such 16-column vectors are to be solved. This is best done by the matrix method. By this method, the matrix of the structure is augmented by the constant column vectors, and also by a check vector on the right (10). This check vector gives a check on calculations at any stage. On the left, the matrix is augmented by an identity matrix  $I_3$ . (See Table 1.)

By elementary row operations, the matrix of the structure

is changed to its canonical form, which happens to be identity matrix. When the same row operations are carried on constant vectors, they are transformed into solution vectors. Each solution vector is the solution for  $x_1$ ,  $x_2$ , and  $x_3$  for the load position considered. When the same row operations are carried out on the augmented identity matrix, it is transformed into the inverse of the matrix (Table 1).

Ordinates required for drawing influence lines for  $H_D$ ,  $V_D$ , and  $M_D$  are shown in Table 2. The influence lines are shown in Fig. (9).

The following values for the matrix of the structure are obtained by use of "Tafel der Werte" (11).

$$f_{11} = +4950$$

$$f_{12} = f_{21} = -2430$$

$$f_{13} = f_{31} = -405$$

$$f_{22} = +2736$$

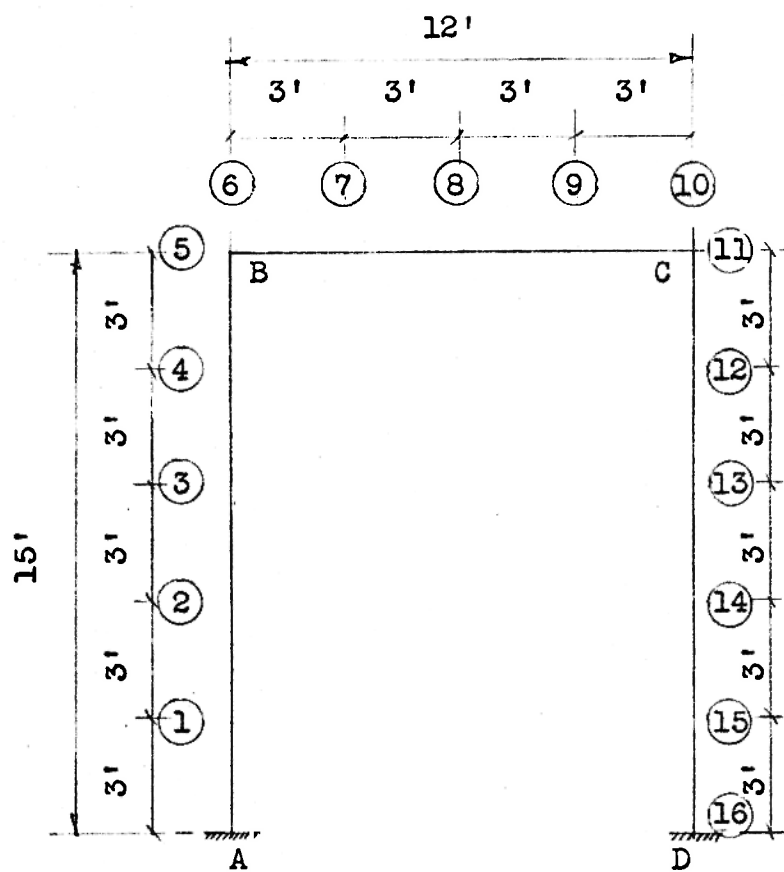
$$f_{23} = f_{32} = +252$$

$$f_{33} = +42$$

The matrix will be

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 4950 & -2430 & -405 \\ -2430 & 2736 & 252 \\ -405 & 252 & 42 \end{bmatrix}$$





(a) Moving Load Positions Marked  
As (1) . . . (16)

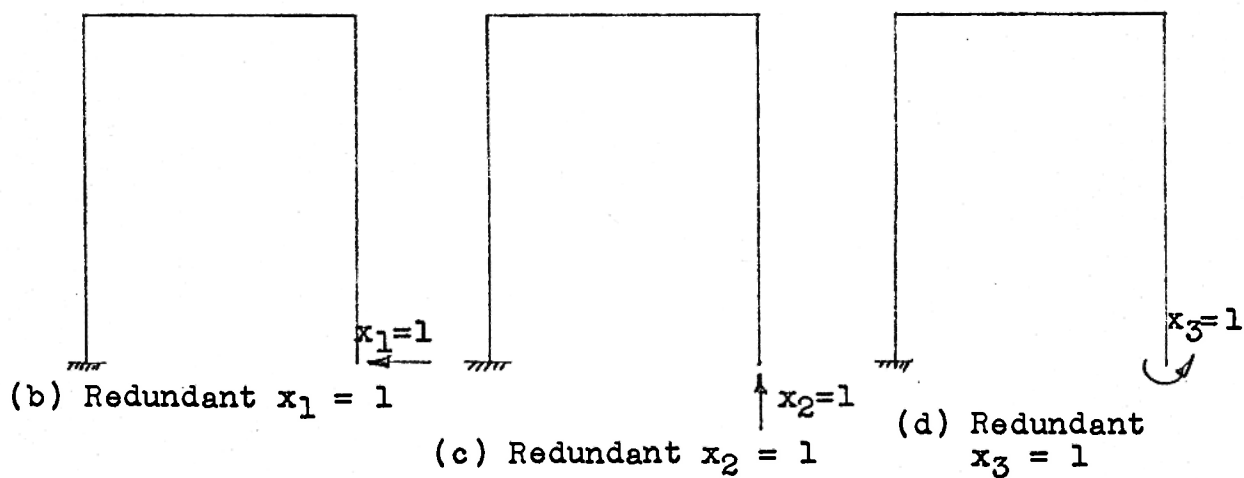


Fig. 7. FRAME WITH REDUNDANTS.

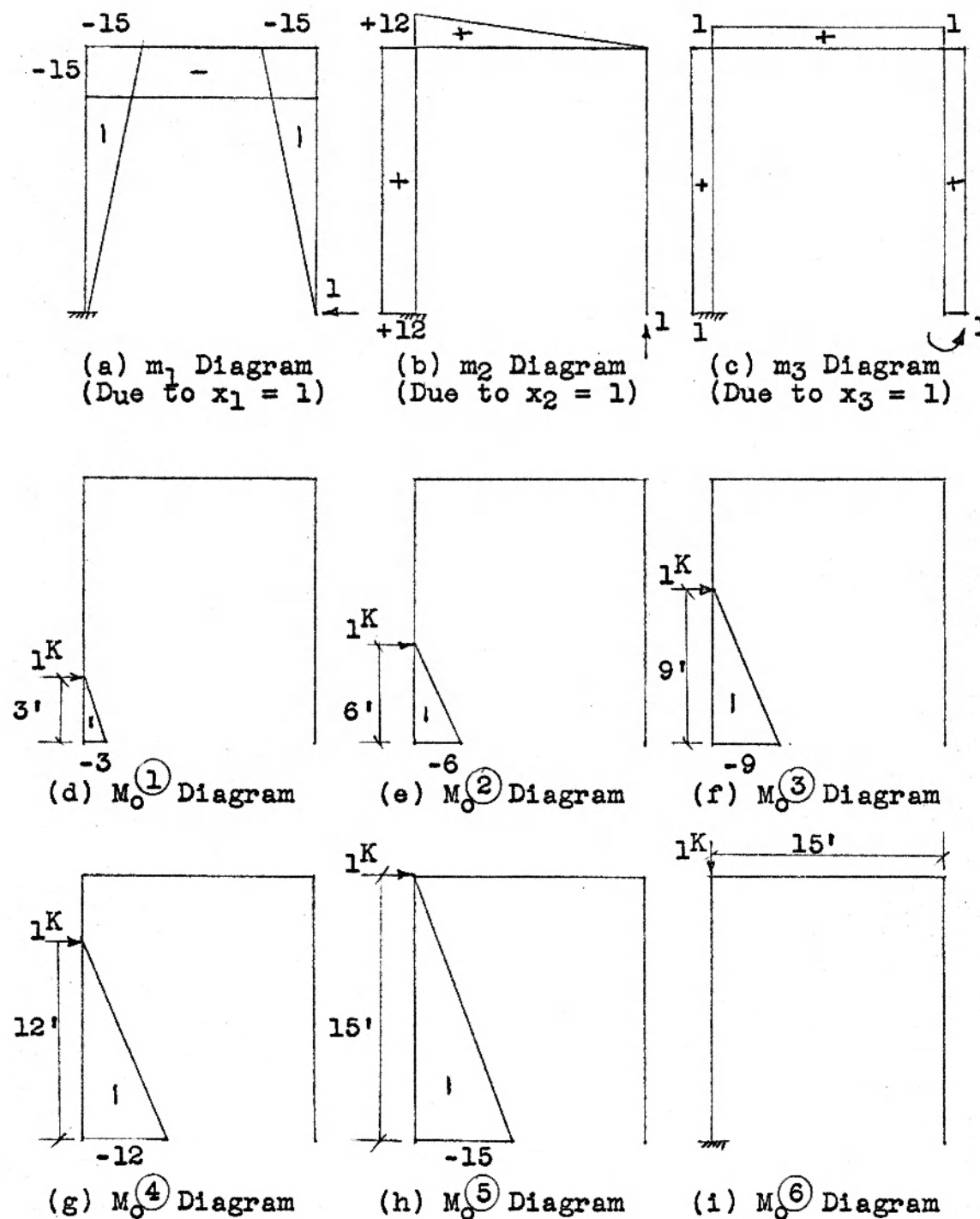


Fig. 8. MOMENT DIAGRAMS DUE TO UNIT VALUE OF REDUNDANTS AND MOVING LOAD.

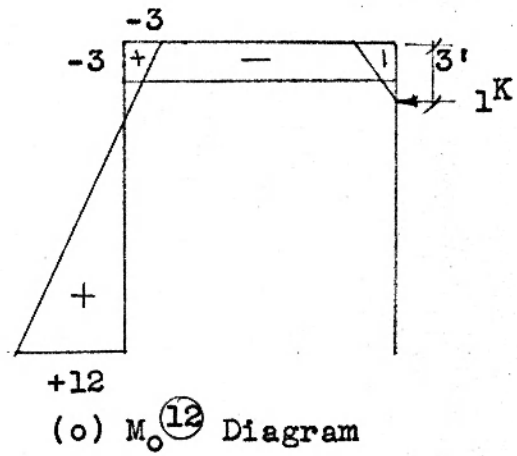
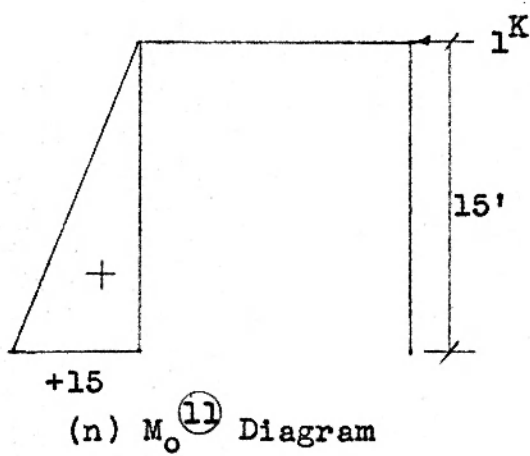
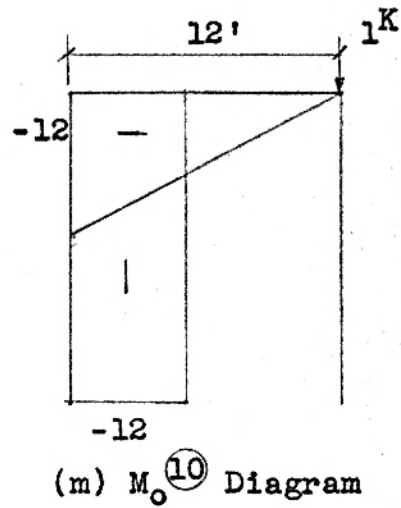
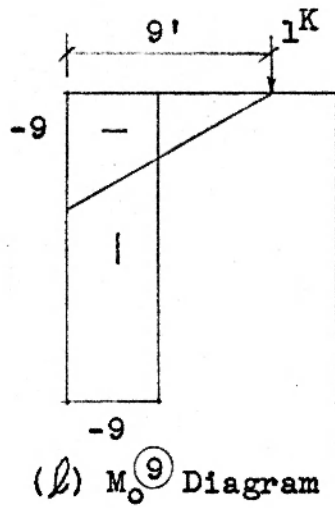
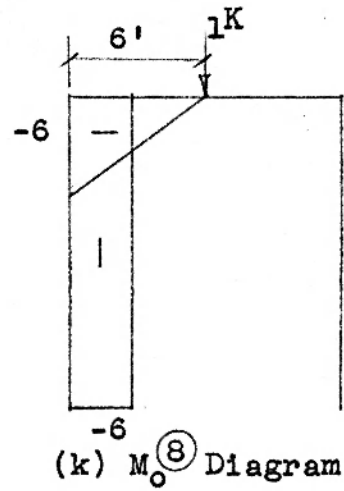
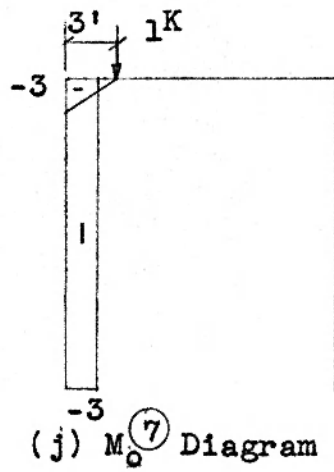


Fig. 8. MOMENT DIAGRAMS DUE TO UNIT VALUE OF REDUNDANTS AND MOVING LOAD.

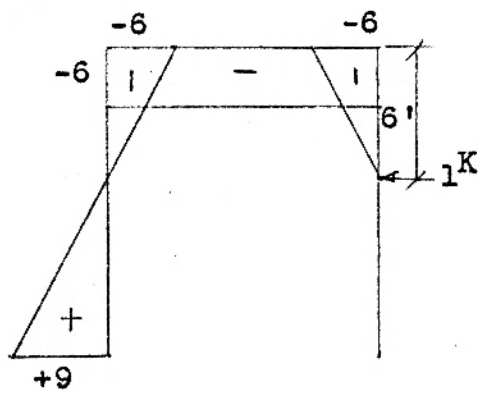
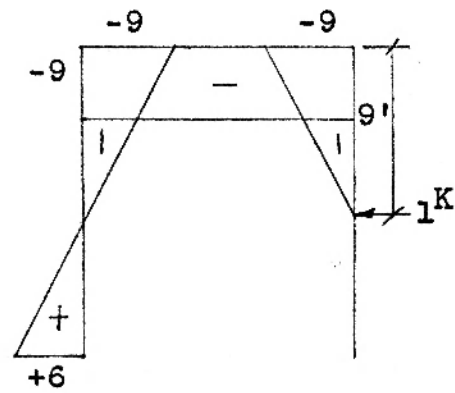
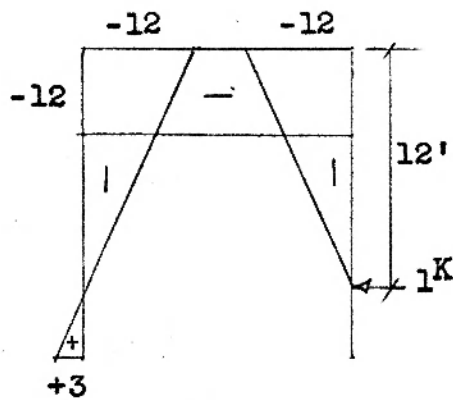
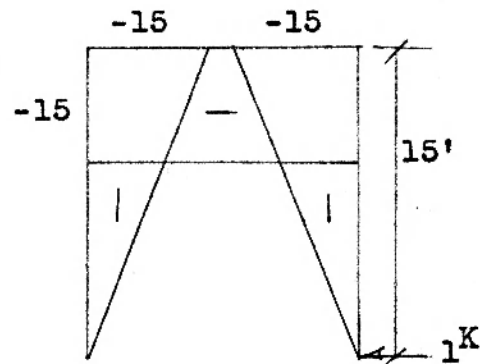
(p)  $M_o^{13}$  Diagram(q)  $M_o^{14}$  Diagram(r)  $M_o^{15}$  Diagram(s)  $M_o^{16}$  Diagram

Fig. 8. MOMENT DIAGRAMS DUE TO UNIT VALUE OF REDUNDANTS AND MOVING LOAD.

Values of constant vectors for the load position indicated:

①	$f_{10}^1 = +4.5$	$f_{20}^1 = -54$	$f_{30}^1 = -4.5$
②	$f_{10}^2 = +36$	$f_{20}^2 = -216$	$f_{30}^2 = -18$
③	$f_{10}^3 = +121.5$	$f_{20}^3 = -486$	$f_{30}^3 = -40.5$
④	$f_{10}^4 = +288$	$f_{20}^4 = -864$	$f_{30}^4 = -72$
⑤	$f_{10}^5 = +562.5$	$f_{20}^5 = -1350$	$f_{30}^5 = -112.5$
⑥	$f_{10}^6 = 0$	$f_{20}^6 = 0$	$f_{30}^6 = 0$
⑦	$f_{10}^7 = +405$	$f_{20}^7 = -589.5$	$f_{30}^7 = -49.5$
⑧	$f_{10}^8 = +895$	$f_{20}^8 = -1260$	$f_{30}^8 = -108$
⑨	$f_{10}^9 = +1620$	$f_{20}^9 = -1984.5$	$f_{30}^9 = -175.5$
⑩	$f_{10}^{10} = +2430$	$f_{20}^{10} = -2736$	$f_{30}^{10} = -252$
⑪	$f_{10}^{11} = -562.5$	$f_{20}^{11} = +1350$	$f_{30}^{11} = +112.5$
⑫	$f_{10}^{12} = +1278$	$f_{20}^{12} = +594$	$f_{30}^{12} = +27$
⑬	$f_{10}^{13} = +2101.5$	$f_{20}^{13} = -162$	$f_{30}^{13} = -67.5$
⑭	$f_{10}^{14} = +3006$	$f_{20}^{14} = -918$	$f_{30}^{14} = -171$
⑮	$f_{10}^{15} = +3964.5$	$f_{20}^{15} = -1674$	$f_{30}^{15} = -283.5$
⑯	$f_{10}^{16} = +4950$	$f_{20}^{16} = -2430$	$f_{30}^{16} = -405$

Table 1. Elementary Row Operations in Finding Out  
Values of Unknown Redundants at Various  
Positions of Moving Load.

Row operations						
$R_1$	4950	-2430	-405	1	0	0
$R_2$	-2430	2736	252	0	1	0
$R_3$	-405	252	42	0	0	1
$R_1$	1	-0.4909	-0.0818	0.0002	0	0
$R_2$	-2430	2736	252	0	1	0
4950	-405	252	42	0	0	1
$R_2 + 2430R_1$	1	-0.4909	-0.0818	0.0002	0	0
$R_3 + 405R_1$	0	1543.113	53.226	0.486	1	0
	0	53.186	8.871	0.081	0	1
$R_2$	1	-0.4909	-0.0818	0.0002	0	0
$R_2$	0	1	0.03449	0.00315	0.000648	0
1543.113	0	53.186	8.871	0.081	0	1
$R_1 +$	1	0	-0.06487	0.000355	0.000318	0
0.4909 $R_2$	0	1	0.03449	0.000315	0.000648	0
$R_3 -$	0	0	7.0366	0.06425	-0.03447	1
53.186 $R_2$						
$R_3$	1	0	-0.06487	0.000355	0.000318	0
$R_3$	0	1	0.03449	0.000315	0.000648	0
7.0366	0	0	1	0.009131	-0.004899	0.14211
$R_1 +$						
0.06487 $R_3$	1	0	0	0.000947	0	0.009219
$R_2 -$	0	1	0	0	0.000817	-0.004901
0.03449 $R_3$	0	0	1	0.009131	-0.004899	0.14211

Table 1 (cont.).

Row	Load positions				
operations	(1)	(2)	(3)	(4)	(5)
$R_1$	4.5	36	121.5	288	562.5
$R_2$	-54.00	-216	-486.0	-864	-1350.0
$R_3$	-4.5	-18	-40.5	-72	-112.5
$R_1$	0.0009	0.0072	0.0243	0.0576	0.1125
	-54	-216	-486	-864	-1350
4950	-4.5	-18	-40.5	-72	-112.5
$R_2+2430R_1$	0.0009	0.0072	0.0243	0.0576	0.1125
$R_3+405R_1$	-51.813	-198.504	-426.951	-724.032	-1076.625
	-4.136	-15.084	-30.159	-48.672	-66.938
$R_2$	0.0009	0.0072	0.0243	0.0576	0.1125
	-0.0335	-0.1286	-0.2767	-0.4692	-0.6977
1543.113	-4.136	-15.084	-30.159	-48.672	-66.938
$R_1 +$	-0.0074	-0.0559	-0.1115	-0.1727	-0.2300
$0.4909R_2$	-0.0335	-0.1286	-0.2767	-0.4692	-0.6977
$R_3-53.186R_2$	-2.3498	-8.2428	-15.4442	-23.719	-29.8326
$R_3$	-0.0074	-0.0559	-0.1115	-0.1727	-0.2300
	-0.0335	-0.1286	-0.2767	-0.4692	-0.6977
7.0366	-0.3339	-1.1714	-2.1948	-3.3707	-4.2395
$R_1+0.06487R_3$	-0.0291	-0.1319	-0.2539	-0.3914	-0.5050
$R_2-0.03449R_3$	-0.0220	-0.0882	-0.2010	-0.3529	-0.5514
	-0.3339	-1.1714	-2.1948	-3.3707	-4.2395

Table 1 (cont.).

Row	:	Load positions								
operations	:	⑥	:	⑦	:	⑧	:	⑨	:	⑩
$R_1$	0	405		895		1620		2430		
$R_2$	0	-589.5		-1260		-1984.5		-2736		
$R_3$	0	-49.5		-108		-175.5		-252		
$R_1$	0	0.0810		0.1790		0.3240		0.4860		
	0	-589.5		-1260		-1984.5		-2736		
4950	0	-49.5		-108		-175.5		-252		
$R_2+2430R_1$	0	0.0810		0.1790		0.3240		0.4860		
$R_3+405R_1$	0	-392.67		-825.03		-1197.18		-1555.02		
	0	-16.695		-35.505		-44.27		-55.17		
$R_2$	0	0.0810		0.1790		0.3240		0.4860		
	0	-0.2545		-0.5346		-0.7758		-1.0076		
1543.113	0	-16.695		-35.505		-44.27		-55.17		
$R_1+0.4909R_2$	0	-0.0439		-0.0834		-0.0568		-0.0086		
$R_3-53.186R_2$	0	-0.2545		-0.5346		0.7758		-1.0076		
	0	-3.1619		-7.071		-3.0103		-1.5776		
$R_3$	0	-0.0439		-0.0834		-0.0568		-0.0086		
	0	-0.2545		-0.5346		-0.7758		-1.0076		
7.0366	0	-0.4493		-1.0049		-0.4278		-0.2242		
$R_1+0.06487R_3$	0	-0.0730		-0.1487		-0.0845		-0.0232		
$R_2-0.03449R_3$	0	-0.2390		-0.5000		-0.7610		-0.9999		
	0	-0.4493		-1.0049		-0.4278		-0.2242		



Table 1 (cont.).

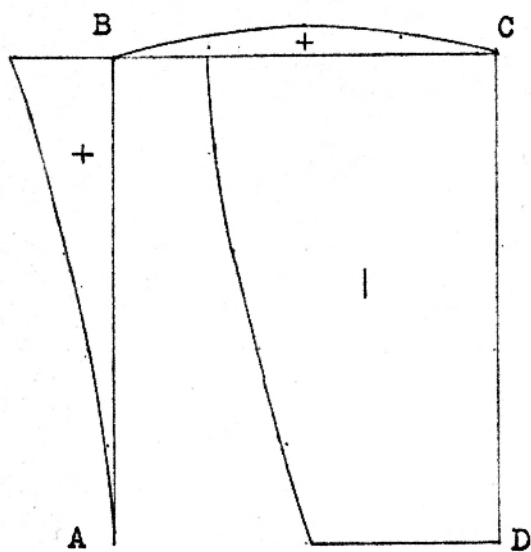
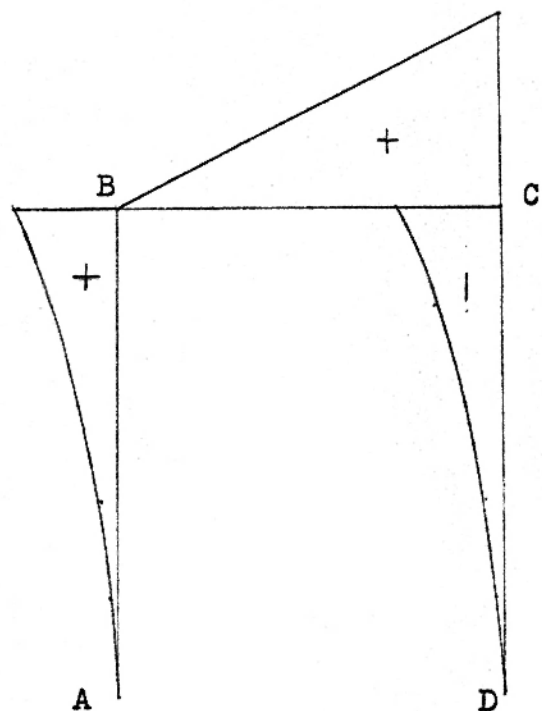
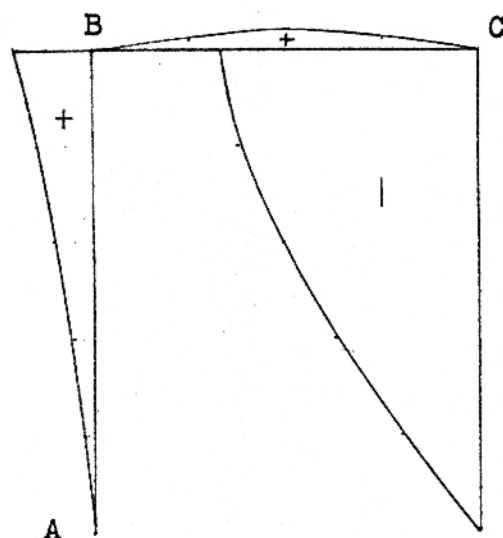
Row	:	Load positions						
operations	:	(11)	:	(12)	:	(13)	:	(14)
$R_1$		-562.5		+1278		2101.5		3006
$R_2$		1350.0		594.0		-162		-918.0
$R_3$		112.5		27.0		-67.50		-171.0
$R_1$		-0.1125		0.2556		0.4203		0.6012
		1350		594		-162		-918
4950		112.5		27		-67.5		-171
$R_2+2430R_1$		-0.1125		0.2556		0.4203		0.6012
$R_3+405R_1$		1076.625		1215.108		859.329		542.916
		66.938		130.518		102.722		72.486
$R_2$		-0.1125		0.2556		0.4203		0.6012
		0.6977		0.7874		0.5569		0.3518
1543.113		66.938		130.518		102.722		72.486
$R_1+0.4909R_2$		0.2300		0.6421		0.6937		0.7739
$R_3-53.186R_2$		0.6977		0.7874		0.5569		0.3518
		29.8327		88.6393		73.1027		53.7752
$R_3$		0.2300		0.6421		0.6937		0.7739
		0.6977		0.7874		0.5569		0.3518
7.0366		4.2395		12.5965		10.3886		7.6420
$R_1+0.06487R_3$		0.5050		1.4592		1.3676		1.2696
$R_2-0.03449R_3$		0.5514		0.3529		0.1985		0.0882
		4.2395		12.5965		10.3886		7.6420

Table 1 (concl.).

Row	Load positions		Check
operations	(15)	(16)	column
$R_1$	3964.5	4950	23216.00
$R_2$	-1674	-2430	-12221.00
$R_3$	-283.5	-405	-1730.00
$R_1$	0.7929	1	4.69010
	-1674	-2430	-12221.0
4950	-283.5	-405	-1730.00
$R_2+2430R_1$	0.7929	1	4.6901
	252.747	0	-824.057
$R_3+405R_1$	37.625	0	169.4905
$R_2$	0.7929	1	4.6901
	0.1638	0	-0.5340
1543.113	37.625	0	169.4905
$R_1+0.4909R_2$	0.8733	1	4.4280
	0.1638	0	-0.5340
$R_3-53.186R_2$	28.9131	0	197.8918
$R_3$	0.8733	1	4.4280
	0.1638	0	-0.5340
7.0366	4.1088	0	28.1232
$R_1+0.06487R_3$	1.1398	1	6.2523
	0.0221	0	-1.5040
$R_2-0.03449R_3$	4.1088	0	28.1232

Table 2. Ordinates for Unknown Redundants for  
Different Positions of Moving Load.

Position : or unit load :	Ordinates for		
	$x_1 = H_D$	$x_2 = V_D$	$x_3 = M_D$
0	0	0	0
①	+0.0291	+0.0220	+0.3339
②	+0.1319	+0.0882	+1.1714
③	+0.2539	+0.2010	+2.1948
④	+0.3914	+0.3529	+3.3707
⑤	+0.5050	+0.5514	+4.2395
⑥	0	0	0
⑦	+0.0730	+0.2390	+0.4493
⑧	+0.1487	+0.5000	+1.0049
⑨	+0.0845	+0.7610	+0.4278
⑩	+0.0232	+0.9999	+0.2242
⑪	-0.5050	-0.5514	-4.2395
⑫	-1.4592	-0.3529	-12.5965
⑬	-1.3676	-0.1985	-10.3886
⑭	-1.2696	-0.0882	-7.6420
⑮	-1.1398	-0.0221	-4.1088
⑯	-1.0000	0	0

(a) Influence Lines for  $H_D$ (b) Influence Lines for  $V_D$ (c) Influence Lines for  $M_D$ Fig. 9. INFLUENCE LINES FOR  $H_D$ ,  $V_D$ , AND  $M_D$ .

### Displacement Method Example

Required: Influence lines for  $\theta_B$  and  $\theta_C$ .

The slope deflection equations are as follows ( $EI = \text{constant}$ ).

$$\begin{aligned} M_{AB} &= \frac{2EI}{\ell} (2\theta_A + \theta_B) - M_{AB}^F \\ &= \frac{2}{15} (0 + \theta_B) - M_{AB}^F \end{aligned} \quad (1)$$

$$\begin{aligned} M_{BA} &= \frac{2EI}{\ell} (2\theta_B + \theta_A) + M_{BA}^F \\ &= \frac{2}{15} (2\theta_B) + M_{BA}^F \end{aligned} \quad (11)$$

$$\begin{aligned} M_{BC} &= \frac{2EI}{\ell} (2\theta_B + \theta_C) - M_{BC}^F \\ &= \frac{2}{12} (2\theta_B + \theta_C) - M_{BC}^F \end{aligned} \quad (111)$$

$$\begin{aligned} M_{CB} &= \frac{2EI}{\ell} (2\theta_C + \theta_B) + M_{CB}^F \\ &= \frac{2}{12} (2\theta_C + \theta_B) + M_{CB}^F \end{aligned} \quad (1v)$$

$$\begin{aligned} M_{CD} &= \frac{2EI}{\ell} (2\theta_C + \theta_D) - M_{CD}^F \\ &= \frac{2}{15} (2\theta_C) - M_{CD}^F \end{aligned} \quad (v)$$

$$M_{DC} = \frac{2EI}{\ell} (2\theta_D + \theta_C) + M_{DC}^F$$

$$= \frac{2}{15} (\theta_C) + M_{DC}^F \quad (vi)$$

Considering joint B,

$$M_{BA} + M_{BC} = 0$$

$$\text{Therefore } \frac{4}{15} \theta_B + \frac{4}{12} \theta_B + \frac{2}{12} \theta_C + (M_{BA}^F - M_{BC}^F) = 0$$

$$\text{Therefore } \frac{3}{5} \theta_B + \frac{1}{6} \theta_C + (M_{BA}^F - M_{BC}^F) = 0 \quad (vii)$$

Considering joint C,

$$M_{CB} + M_{CD} = 0$$

$$\text{Therefore } \frac{4}{12} \theta_C + \frac{4}{15} \theta_C + \frac{2}{12} \theta_B + (M_{CB}^F - M_{CD}^F) = 0$$

$$\text{Therefore } \frac{1}{6} \theta_B + \frac{3}{5} \theta_C + (M_{CB}^F - M_{CD}^F) = 0 \quad (viii)$$

Putting in matrix form,

$$\begin{bmatrix} 3/5 & 1/6 \\ 1/6 & 3/5 \end{bmatrix} \begin{bmatrix} EI \theta_B \\ EI \theta_C \end{bmatrix} = - \begin{bmatrix} M_{BA}^F - M_{BC}^F \\ M_{CB}^F - M_{CD}^F \end{bmatrix}$$

$$BX = F_0$$

where

$$B = \begin{bmatrix} 3/5 & 1/6 \\ 1/6 & 3/5 \end{bmatrix} = \text{stiffness matrix}$$

$$X = \begin{bmatrix} EI \theta_B \\ EI \theta_C \end{bmatrix} = \text{unknown vector}$$

$$F_0 = \begin{bmatrix} M_{BC}^F - M_{BA}^F \\ M_{CD}^F - M_{CB}^F \end{bmatrix} = \text{column vector}$$

$$B = \begin{bmatrix} 3/5 & 1/6 \\ 1/6 & 3/5 \end{bmatrix} \quad \text{Adj } B = \begin{bmatrix} 3/5 & -1/6 \\ -1/6 & 3/5 \end{bmatrix}$$

$$|B| = 3/5 \times 3/5 - [(-1/6)(-1/6)]$$

$$= \frac{9}{25} - \frac{1}{36} = \frac{299}{900}$$

$$\text{Therefore } B^{-1} = \frac{\text{adj } B}{|B|} = \frac{\begin{bmatrix} 3/5 & -1/6 \\ -1/6 & 3/5 \end{bmatrix}}{\frac{299}{900}} = \begin{bmatrix} \frac{540}{299} & -\frac{150}{299} \\ -\frac{150}{299} & \frac{540}{299} \end{bmatrix}$$

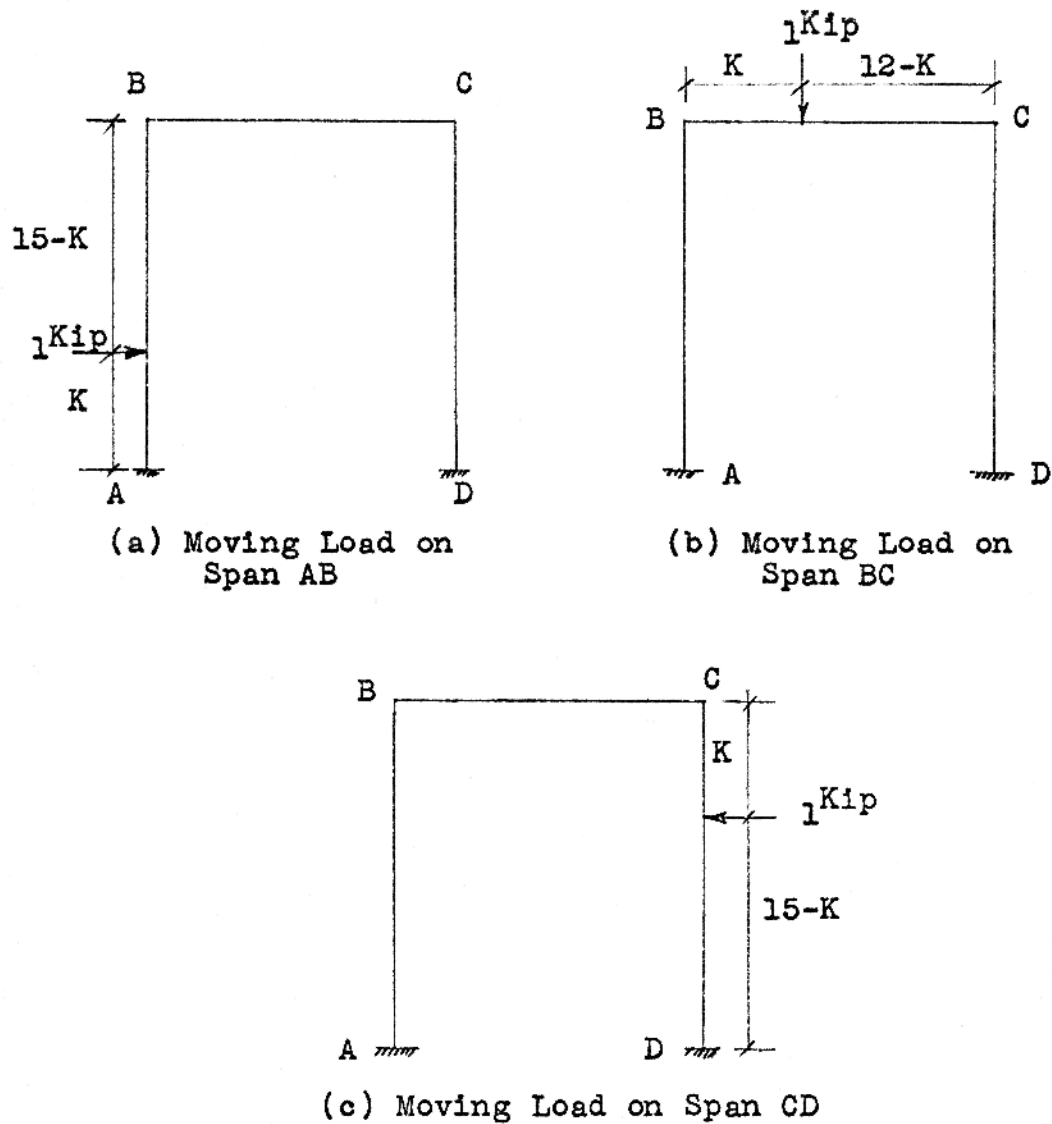


Fig. 10. FRAME WITH MOVING LOAD ON DIFFERENT SPANS.



Table 3. Fixed-end Moments Due to Different Load Positions.

1	2	3	4	5	6
Load position	Distance 'K'	$M_{AB}^F = \frac{1K(15-K)^2}{225}$ in ft-kips	$M_{BA}^F = \frac{1K(15-K)^2}{225}$ in ft-kips	$M_{BC}^F = \frac{1K(12-K)^2}{144}$ in ft-kips	$M_{CB}^F = \frac{1K(12-K)^2}{144}$ in ft-kips
Load in Span AB:					
0	0	0	0	0	0
①	3	1.92	0.48	0	0
②	6	2.16	1.44	0	0
③	9	1.44	2.16	0	0
④	12	0.48	1.92	0	0
⑤	15	0	0	0	0
Load in Span BC:					
⑥	0	0	0	0	0
⑦	3	0	0	1.6875	0.5625
⑧	6	0	0	1.5000	1.5000
⑨	9	0	0	0.5625	1.6875
⑩	12	0	0	0	0
Load in Span CD:					
⑪	0	0	0	0	0
⑫	3	0	0	0	0
⑬	6	0	0	0	0
⑭	9	0	0	0	0
⑮	12	0	0	0	0
⑯	15	0	0	0	0

Table 3 (concl.).

1	2	7	8	9	10
Load : Dis- : $M_{CD}^F =$ : $M_{DC}^F =$ : $M_{BC}^F - M_{BA}^F$ : $M_{CD}^F - M_{CB}^F$	posi- : tance : $\frac{1K(15-K)^2}{225}$ : $\frac{1K(15-K)^2}{225}$ : in ft-kips : in ft-kips	tion : 'K' : in ft-kips : in ft-kips : : :			
Load in Span AB:					
0	0	0	0	0	0
①	3	0	0	-0.48	0
②	6	0	0	-1.44	0
③	9	0	0	-2.16	0
④	12	0	0	-1.92	0
⑤	15	0	0	0	0
Load in Span BC:					
⑥	0	0	0	0	0
⑦	3	0	0	1.6875	-0.5625
⑧	6	0	0	1.5000	-1.5000
⑨	9	0	0	0.5625	-1.6875
⑩	12	0	0	0	0
Load in Span CD:					
⑪	0	0	0	0	0
⑫	3	1.92	0.48	0	1.92
⑬	6	2.16	1.44	0	2.16
⑭	9	1.44	2.16	0	1.44
⑮	12	0.48	1.92	0	0.48
⑯	15	0	0	0	0

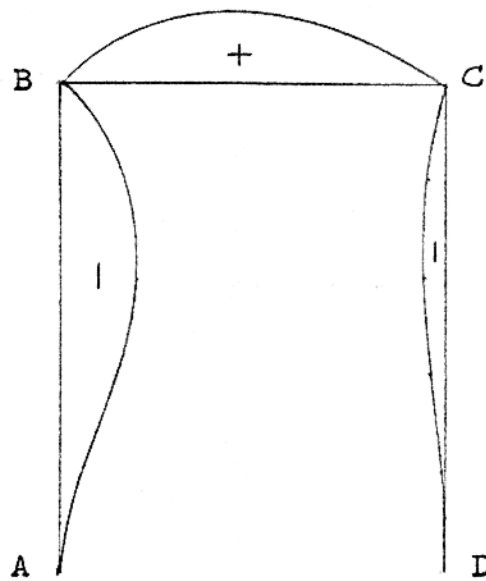
Table 4. Ordinates of  $EI\theta_B$  and  $EI\theta_C$  at Different Load Positions<sup>1</sup>.

Position of load :	$F_0 = \begin{bmatrix} M_{BC}^F & -M_{BA}^F \\ M_{CD}^F & -M_{CB}^F \end{bmatrix}$	$X = B^{-1} F_0 = \begin{bmatrix} EI\theta_B \\ EI\theta_C \end{bmatrix}$	$EI\theta_B$	$EI\theta_C$
0	0	0	0	0
①	$\begin{bmatrix} -0.48 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -0.8669 \\ +0.2408 \end{bmatrix}$	-0.8669	+0.2408
②	$\begin{bmatrix} -1.44 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -2.6006 \\ +0.7225 \end{bmatrix}$	-2.6006	+0.7225
③	$\begin{bmatrix} -2.16 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -3.9010 \\ +1.0837 \end{bmatrix}$	-3.9010	+1.0837
④	$\begin{bmatrix} -1.92 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -3.4675 \\ +0.9633 \end{bmatrix}$	-3.4675	+0.9633
⑤	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	0
⑥	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	0
⑦	$\begin{bmatrix} 1.6875 \\ -0.5625 \end{bmatrix}$	$\begin{bmatrix} 3.3298 \\ -1.8625 \end{bmatrix}$	+3.3298	-1.8625
⑧	$\begin{bmatrix} 1.5000 \\ 1.5000 \end{bmatrix}$	$\begin{bmatrix} 3.4616 \\ -3.4616 \end{bmatrix}$	+3.4616	-3.4616
⑨	$\begin{bmatrix} 0.5625 \\ -1.6875 \end{bmatrix}$	$\begin{bmatrix} +1.8625 \\ -3.3298 \end{bmatrix}$	+1.8625	-3.3298
⑩	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	0
⑪	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	0
⑫	$\begin{bmatrix} 0 \\ 1.92 \end{bmatrix}$	$\begin{bmatrix} -0.9633 \\ +3.4675 \end{bmatrix}$	-0.9633	+3.4675
⑬	$\begin{bmatrix} 0 \\ 2.16 \end{bmatrix}$	$\begin{bmatrix} -1.0837 \\ +3.9010 \end{bmatrix}$	-1.0837	+3.9010

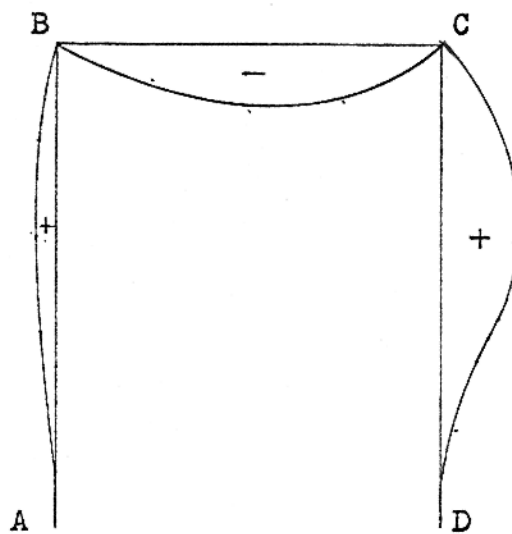
Table 4 (concl.).

Position of load :	$F_0 = \begin{bmatrix} M_{BC}^F & -M_{BA}^F \\ M_{CD}^F & -M_{CB}^F \end{bmatrix}$	$X = B^{-1} F_0 = \begin{bmatrix} EI\theta_B \\ EI\theta_C \end{bmatrix}$	$EI\theta_B$	$EI\theta_C$
⑭	$\begin{bmatrix} 0 \\ 1.44 \end{bmatrix}$	$\begin{bmatrix} -0.7225 \\ +2.6006 \end{bmatrix}$	-0.7225	+2.6006
⑮	$\begin{bmatrix} 0 \\ 0.48 \end{bmatrix}$	$\begin{bmatrix} -0.2408 \\ +0.8669 \end{bmatrix}$	-0.2408	+0.8669
⑯	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	0

<sup>1</sup>The influence lines for  $EI\theta_B$  and  $EI\theta_C$  are shown in Fig. 11.



(a) Influence Lines for  $EI\theta_B$



(b) Influence Lines for  $EI\theta_C$

Fig. 11. INFLUENCE LINES FOR  $EI\theta_B$  AND  $EI\theta_C$ .

## CONCLUSIONS

It is seen that for the structure analyzed, in terms of the matrix formulation, the displacement method has a great advantage over the force method. The advantage of the matrix formulation lies in the fact that all other similar problems may be treated by an efficient standardized procedure. Moreover, the mathematical operations necessary after the original matrices have been established are of such routine nature that they can be carried out by an electronic digital computer, or by persons having no knowledge of structural analysis. A major drawback to the application of high-speed digital computers is, however, the initial difficulty of getting a given type of problem set up and coded.

The extension of the method described here to structures with variable transverse section is feasible and practical. The application of this technique to the solution of Civil Engineering problems presents no greater difficulty than those already surmounted for the analysis of aircraft structures.

## ACKNOWLEDGMENT

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MATRIX SOLUTION FOR INFLUENCE LINES

by

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B. S., Gujarat University, India, 1958

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The main advantage of the matrix approach is the method of notation and the ease of speaking about general problems. The equations for plane redundant structures obtained by the displacement and force method are formulated in the matrix form. The development of the high-speed electronic digital computer inspired structural engineers to analyze highly indeterminate structures without much labor. The matrix formulation is a systematic procedure to be adopted in the programming of the solution.

There are two principal methods of attack for analyzing a linear statically indeterminate structure. The first method involves the determination of certain redundant forces or moments by solving the elastic compatibility equations. The flexibility matrix consisting of an ordered array of the flexibility influence coefficients or deflection influence coefficients is used in this method, which is known as the "force method" of analysis. The second method of analysis, which is very similar to the force method, is known as the "displacement method". This method, however, assumes the joint rotations or displacements as unknowns and involves the solution of the simultaneous joint equilibrium equations. These equations relate the redundant forces in terms of the assumed deflections. They are represented by the stiffness matrix composed of an ordered array of the stiffness influence coefficients.

While discussing these methods, the concepts of the first and second theorems of Castigliano, the theorems of virtual work and Maxwell's law of reciprocal deflections are used.

As many structures are subjected to moving loads, it is essential to understand the methods by which the position of live load which causes the maximum stress at any point may be determined. This is conveniently done and shown by influence lines. In showing for a particular section the variation in shear, moment, reaction, or other direct function due to a unit load moving across the structure, the influence lines are constructed by placing a unit load at various points on a structure. The different ordinates for a few controlling points, required for drawing influence lines corresponding to the different positions of a moving load, are shown in the form of a matrix. The matrix solution for final results, that is, the different ordinates required for influence lines, is thus a convenient, systematic, and efficient method.